Ma/CS 6c

Assignment #8

Due Wednesday, May 28 at 11:59 pm.

(10%) **1.** Find formulas in prenex normal form logically equivalent to the following (here P, Q, R are relation symbols).

(a) $\neg\exists x[\forall z P(x,y,z) \Rightarrow \neg\exists y(Q(y) \wedge R(z))]$

(b) $\neg\forall x\forall y[R(z) \Rightarrow \neg\forall z\forall w(P(x,u,z) \vee \exists u\neg Q(u,w))]$

(30%) **2.** Consider the language $\mathcal{L} = \{a, +, \cdot, <\}$, where a is a constant symbol, $+, \cdot$ are binary function symbols and $<$ is a binary relation symbol, and its structure

$$\mathcal{M} = \langle M, 2, +, \cdot, < \rangle$$

where $M = \{2, 3, 4, \dots\}$ and $+, \cdot, <$ have their usual meanings. Let

$$\begin{aligned} A : \forall x\exists y\forall z\exists w\exists v\{(x < y) \wedge [z = w \cdot v \\ \vee\, w \cdot z = (x + y) \\ \vee\, w \cdot (x + y) = v \cdot z + a]\}. \end{aligned}$$

Describe $G_A^{\mathcal{M}}$ and determine which of the two players $\exists, \forall$ has a winning strategy.

(20%) **3.** Show that

(a) $\vdash \forall x(P(x) \wedge Q(x)) \Leftrightarrow (\forall x P(x) \wedge \forall x Q(x))$

(b) $\vdash \exists x(P(x) \Rightarrow \forall x P(x))$

(c) $\{P(x), \forall y(P(y) \Rightarrow \forall z Q(z))\} \vdash \forall x Q(x)$

Here P, Q are unary relation symbols. *You are not supposed to use the Completeness Theorem.* Recall that formulas with $\wedge$ or $\Leftrightarrow$ are abbreviations for formulas whose only connectives are $\neg, \Rightarrow$.

(20%) **4.** Consider the language $L = \{R\}$, where R is a binary relation symbol, and the set of axioms

$$S = \{\forall x\neg R(x,x), \forall x\forall y(R(x,y) \Rightarrow R(y,x))\}$$

in this language. A model $\mathcal{G} = \langle G, R^{\mathcal{G}} \rangle$ of these axioms is called an (undirected) *graph*. If $\mathcal{G}$ is such a graph and $a, b \in G$ are two vertices, we say that the *distance between* a, b *is* $\leq n$ if there are vertices $a_0 = a, a_1, a_2, \dots, a_m = b$ such that $m \leq n$ and $R^{\mathcal{G}}(a_i, a_{i+1})$ for $i = 0, \dots, m-1$, i.e., a_i, a_{i+1} are connected by an edge.

(a) Write a formula $A_n(x, y)$ of L such that for each graph $\mathcal{G}$ and each $a, b \in G$

$$\mathcal{G} \models A_n[a, b] \text{ iff the distance between } a, b \text{ is } \leq n.$$

(b) A graph $\mathcal{G}$ is *connected* if for each $a, b \in G$ there is a path from a to b i.e., a sequence $a_1 = a, a_1, \ldots, a_n = b$ such that $R^{\mathcal{G}}(a_i, a_{i+1})$ for $i = 0, \ldots, n-1$. Show that there is no set of sentences S in L such that

$$\mathcal{G} \models S \text{ iff } \mathcal{G} \text{ is a connected graph}$$

(i.e., connectedness cannot be expressed by axioms in first-order logic).

(20%) **5.** Let $\mathcal{N} = \langle \mathbb{N}, 0, S, +, \cdot, < \rangle$ be the standard structure of arithmetic and

$$T = Th(\mathcal{N}) = \{A : \mathcal{N} \models A, A \text{ a sentence}\}.$$

Let P be any set of prime numbers. Show that there is a model

$$\mathcal{M} = \langle M, \ldots \rangle \models T,$$

and $a \in M$ such that for any $p \in P$ we have

$$\mathcal{M} \models D_p[a]$$

and for any prime $p \notin P$ we have

$$\mathcal{M} \models \neg D_p[a],$$

where $D_p(x)$ is the formula

$$\exists y(\underline{p} \cdot y = x),$$

with $\underline{p} = \underbrace{S(S(S \ldots S}_{p \text{ times}}(0) \ldots))$.