Ma/CS 6c

Assignment #7

Due Wednesday, May 21 at 1:00 pm.

(20%) **1.** Call a set $X \subseteq \mathbb{N}$ *eventually periodic* if there is $n_0 \in \mathbb{N}$ and $p \in \mathbb{N}, p \geq 1$ (a *period*) such that for all $n \geq n_0$,

$$n \in X \text{ iff } n + p \in X.$$

Show that every eventually periodic set is first-order definable in the structure

$$\mathcal{N}' = \langle \mathbb{N}; 0, S, + \rangle.$$

(It turns out that these are the *only* first-order definable sets in $\mathcal{N}'$.) We of course identify here subsets of $\mathbb{N}$ with unary relations on $\mathbb{N}$. The language here is $L' = \{0; S, +\}$.

(20%) **2.** Show that $+$ is *not* first-order definable in the structure

$$\mathcal{N}'' = \langle \mathbb{N}, \cdot \rangle.$$

The language here is $L = \{\cdot\}$.

(20%) **3.** Consider the language $L = \{f, g\}$ with f, g unary function symbols, and its structure

$$\mathcal{M} = \langle \mathbb{Z} \times \mathbb{Z}, f^{\mathcal{M}}, g^{\mathcal{M}} \rangle,$$

where

$$f^{\mathcal{M}}(i, j) = (i, j + 1)$$
$$g^{\mathcal{M}}(i, j) = (i + 1, j).$$

Determine all the first-order definable subsets (i.e., unary relations) of $M = \mathbb{Z} \times \mathbb{Z}$.

(20%) **4.** Consider the language $L = \{<\}$ and it structure

$$\mathcal{A} = \langle \mathbb{R}, < \rangle.$$

Determine all first-order definable unary (i.e., subsets of $\mathbb{R}$) and binary (i.e., subsets of $\mathbb{R}^2$) relations in this structure.

(20%) **5.** Let L be a first-order language, and A a sentence of L. The *spectrum* of A is the set of cardinalities of *finite* models of A, i.e.,

$$\text{spectrum}(A) = \{n \geq 1 : \text{ there is a model } \mathcal{M} = \langle M, - \rangle \text{ of } A \\ \text{with cardinality}(M) = n\}.$$

So $\text{spectrum}(A) \subseteq \mathbb{N} \backslash \{0\}$. For each of the sets X below find L, A so that $\text{spectrum}(A) = X$:

(i) $X = \emptyset$;

(ii) $X = \mathbb{N} \setminus \{0\}$;

(iii) $X = \{n \geq 1 : n \text{ is even}\}$;

(iv) $X = \{n \geq 1 : n \text{ is a square}\}$.