Ma/CS 6c
Assignment #6
Due Wednesday, May 14 at 1:00 pm.

(20%) **1.** Prove that a proper initial segment of a formula in first-order logic is *not* a formula.

(10%) **2.** (a) Write a sentence A_6 in the language with no non-logical symbols (but recall that "=" is available), such that for any structure $\mathcal{M} = \langle M \rangle$ for this language:

$$\mathcal{M} \models A_6 \text{ iff } M \text{ has exactly 6 elements.}$$

(b) Write a sentence B in the language $L = \{R\}$, R a binary relation symbol, such that for any structure $\mathcal{M} = \langle M, R^{\mathcal{M}} \rangle$ for this language, we have:

$$\mathcal{M} \models B \text{ iff } R^{\mathcal{M}} \text{ is the graph of a one-to-one correspondence of } M \text{ with itself (i.e., a permutation of } M).$$

(10%) **3.** Find a sentence A in the language $L = \{<\}$, $<$ a binary relation symbol, such that

$$\langle \mathbb{Q}, < \rangle \models A$$

but

$$\langle \mathbb{Z}, < \rangle \models \neg A.$$

(Here $\mathbb{Q}$ = rationals, $\mathbb{Z}$ = integers, and $<$ is the usual ordering in each one of them.)

(10%) **4.** Write a sentence A in the language $L = \{R\}$, R a binary relation symbol, such that for any structure $\mathcal{M} = \langle M, R^{\mathcal{M}} \rangle$ for L we have:

$$\mathcal{M} \models A \text{ iff } R^{\mathcal{M}} \text{ is an equivalence relation on } M \text{ which has exactly 4 equivalence classes each of which has exactly 8 elements.}$$

(20%) **5.** Consider the language $L = \{P\}$, P a binary relation symbol, and the sentences

$$A_1 : \forall x \exists y P(x, y)$$

$$A_2 : \forall x \forall y \forall z [P(x, y) \wedge P(y, z) \Rightarrow P(x, z)]$$

$$A_3 : \forall x \neg P(x, x)$$

Assume that $\mathcal{M} = \langle M, P^{\mathcal{M}} \rangle \models A_1 \wedge A_2 \wedge A_3$. What can you say about the cardinality of M?

(10%) **6.** Show that the formula

$$(\exists x A \wedge \exists x B) \Rightarrow \exists x (A \wedge B) \qquad (*)$$

is in general not logically valid (i.e., find a language L and formulas A, B in L such that the formula $(*)$ is not logically valid).

(10%) **7.** Show that if S is a binary relation symbol, then

$$\models \neg \exists y \forall x (S(y,x) \Leftrightarrow \neg S(x,x)).$$

(10%) **8.** Let $\mathcal{M} = \langle \mathbb{N}, < \rangle$, where $<$ is the usual ordering on $\mathbb{N}$, and let $\mathcal{S} = \langle \mathbb{N}, <^{\mathcal{S}} \rangle$, where $<^{\mathcal{S}}$ is the following binary relation on $\mathbb{N}$:

$$n <^{\mathcal{S}} m \text{ iff } (n, m \text{ are both even or both odd, and } n < m) \text{ or } (n \text{ is even and } m \text{ is odd}).$$

Find a sentence A in the language $L = \{<\}$, where $<$ is a binary relation symbol, such that

$$\mathcal{M} \models A \text{ but } \mathcal{S} \models \neg A.$$