Math/CS 6c

Assignment #4

Wednesday April 30th at 1:00 pm.

(35%) **1.** Study the attached handout describing an algorithm for transforming a wff A to a cnf formula B, so that

$$A \text{ is satisfiable iff } B \text{ is satisfiable},$$

and prove the correctness of this algorithm.

(30%) **2.** (i) *Using resolution*, show that $p \wedge q \wedge r$ is logically implied by the following set of formulas:

$$\{p \Rightarrow q,\ q \Rightarrow r,\ r \Rightarrow p,\ p \vee q \vee r\}.$$

(ii) *Using resolution*, show that

$$(\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg r) \vee (q \wedge r) \vee p$$

is a tautology.

(35%) **3.** A formula A in conjunctive normal form is called a *Horn formula* if every conjunct of A contains at most one propositional variable (but can contain any number of negated propositional variables). For example,

$$(p \vee \neg q) \wedge (\neg q \vee \neg p \vee r),$$

$$\neg q \wedge \neg p,$$

are Horn formulas, but

$$(p \vee \neg q) \wedge (\neg q \vee \neg p \vee r \vee s)$$

is *not*.

If we write $q_1 \wedge \cdots \wedge q_n \rightarrow 0$ instead of $\neg q_1 \vee \cdots \vee \neg q_n$, $1 \rightarrow q$ instead of q, and $q_1 \wedge \cdots \wedge q_n \rightarrow q$ instead of $\neg q_1 \vee \cdots \vee \neg q_n \vee q$, then the Horn formulas are just the conjunctions of formulas of the form:

$$q_1 \wedge \cdots \wedge q_n \rightarrow 0,$$

$$q_1 \wedge \cdots \wedge q_n \rightarrow q,$$

$$1 \rightarrow q,$$

where $q_1, q_2, \ldots, q_n, q$ are propositional variables.

(i) Write the above two examples of Horn formulas in that form.

(ii) Consider the following efficient algorithm for testing satisfiability of a Horn formula A:

(a) First, mark all the propositional variables p such that $1 \to p$ appears in A.

(b) Next, for any conjunct of the form $q_1 \wedge \cdots \wedge q_n \to q$ with *all* $q_1, \ldots, q_n$ already marked, mark q. Repeat this as long as there are such conjuncts with $q_1, \ldots, q_n$ marked at an earlier stage.

If after doing all that we have exhausted all the conjuncts, output:

"A is satisfiable".

(c) Otherwise, if there is a conjunct of the form $q_1 \wedge \cdots \wedge q_n \to 0$, where $q_1, \ldots, q_n$ have been already marked, then output:

"A is unsatisfiable";

otherwise output again:

"A is satisfiable".

Prove the correctness of this algorithm.

(iii) Apply it to the following examples:

$$A_1 = (p \wedge q \wedge r \to 0) \wedge (s \to 0) \wedge (r \to p) \wedge (1 \to r) \wedge (1 \to q) \wedge (u \to s) \wedge (1 \to u),$$
$$A_2 = (p \wedge q \wedge t \to 0) \wedge (t \to 0) \wedge (r \to p) \wedge (1 \to r) \wedge (1 \to q) \wedge (u \to s) \wedge (1 \to u).$$

Handout
Transformation of a formula A to a formula B in conjunctive normal form, so that A is satisifable iff B is satisfiable.

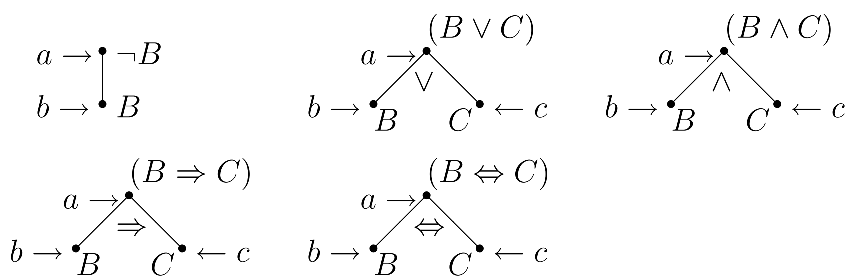
Given a formula A, consider its parse tree T_A. To each node a of T_A associate a propositional variable p_a, so that:

i) If a is terminal, then $p_a = p$, where p is the propositional variable occurring at a;

ii) If a is a non-terminal node, p_a is different from all the propositional variables occurring in A;

iii) To distinct nodes we assign distinct variables.

For each node a which is not terminal, we have one of the following five possibilities



In the first case, associate to a the formula:

$$p_a \Leftrightarrow \neg p_b \tag{1}$$

In the second, associate to a the formula:

$$p_a \Leftrightarrow (p_b \vee p_c) \tag{2}$$

In the third case, the formula:

$$p_a \Leftrightarrow (p_b \wedge p_c) \tag{3}$$

In the fourth case, the formula

$$p_a \Leftrightarrow (p_b \Rightarrow p_c) \tag{4}$$

and in the fifth, the formula

$$p_a \Leftrightarrow (p_b \Leftrightarrow p_c) \tag{5}$$

Now (1) is equivalent to
(1′) $(\neg p_a \vee \neg p_b) \wedge (p_a \vee p_b)$;
(2) is equivalent to
(2′) $(\neg p_a \vee p_b \vee p_c) \wedge (\neg p_b \vee p_a) \wedge (\neg p_c \vee p_a)$;

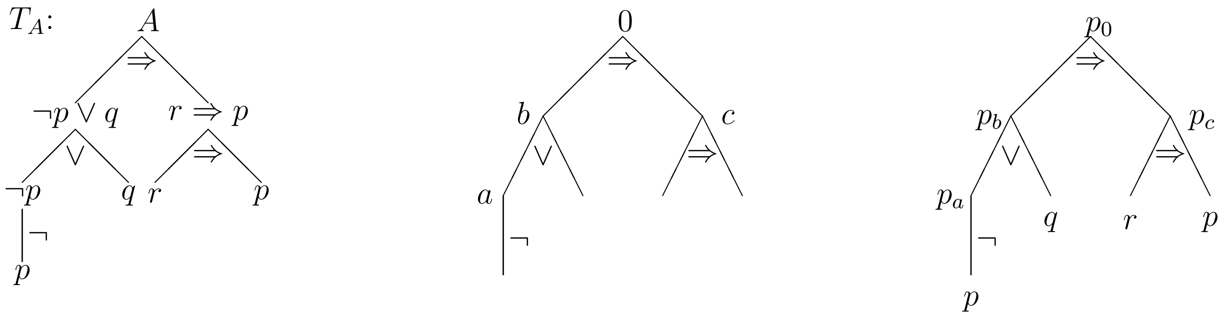
(3) is equivalent to
(3′) $(\neg p_a \vee p_b) \wedge (\neg p_a \vee p_c) \wedge (\neg p_b \vee \neg p_c \vee p_a)$;
(4) is equivalent to
(4′) $(\neg p_a \vee \neg p_b \vee p_c) \wedge (p_a \vee p_b) \wedge (p_a \vee \neg p_c)$;
and (5) to
(5′) $(\neg p_a \vee \neg p_b \vee p_c) \wedge (\neg p_a \vee p_b \vee \neg p_c) \wedge (p_a \vee \neg p_b \vee \neg p_c) \wedge (p_a \vee p_b \vee p_c)$.

Letting 0 be the root of the parse tree of A, define B to be the formula which is the conjunction of p_0 and each of the formulas of type $(1') - (5')$ corresponding to each node of the parse tree of A. Then A is satisfiable iff B is satisfiable.

Notice that each conjunct of B has at most 3 literals.

Example: $A = (\neg p \vee q) \Rightarrow (r \Rightarrow p)$

T_A:



$$
\begin{aligned}
B' &= p_0 \wedge (p_0 \Leftrightarrow (p_b \Rightarrow p_c)) \wedge (p_b \Leftrightarrow p_a \vee q) \wedge (p_c \Leftrightarrow (r \Rightarrow p)) \wedge (p_a \Leftrightarrow \neg p), \\
B &= p_0 \wedge (\neg p_0 \vee \neg p_b \vee p_c) \wedge (p_0 \vee p_b) \wedge (p_0 \vee \neg p_c) \wedge (\neg p_b \vee p_a \vee q) \wedge (\neg p_a \vee p_b) \wedge \\
&\quad \wedge (\neg q \vee p_b) \wedge (\neg p_c \vee \neg r \vee p) \wedge (p_c \vee r) \wedge (p_c \vee \neg p) \wedge (\neg p_a \vee \neg p) \wedge (p_a \vee p).
\end{aligned}
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