Ma/CS 6c

Assignment #2

Due Tuesday, April 15, at 1:00 pm.

(30%) **1.** (i) Let A be a wff and assume that the only connectives appearing in A are among $\neg, \wedge, \vee$ (i.e., $\Rightarrow, \Leftrightarrow$ don't appear). Let A^* be obtained from A by replacing each propositional variable p appearing in A by $\neg p$ and replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. Show that

$$\neg A \equiv A^*$$

(i.e., $\neg A, A^*$ are logically equivalent).

(ii) Suppose A is a wff as in (i). Let A' be the wff obtained from A by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. We call A' the *dual* of A. (*Example*: $(p \vee q) \wedge \neg r$ is the dual of $(p \wedge q) \vee \neg r$.) Show that A is a tautology iff $\neg A'$ is a tautology.

(iii) (*Principle of duality*) For A, B wff as in (ii), show that

$$A \equiv B \text{ iff } A' \equiv B'.$$

(20%) **2.** Consider the wff

$$A_n = ((\ldots((p_1 \Leftrightarrow p_2) \Leftrightarrow p_3) \Leftrightarrow \ldots) \Leftrightarrow p_n).$$

Show that a valuation v satisfies A_n exactly when $v(p_i) = 0$ for an even number of i in the interval $1 \leq i \leq n$.

(20%) **3.** For each $n = 2, 3, 4, \ldots$ find a set $S = \{A_1, A_2, \ldots, A_n\}$ consisting of n wff such that S is *not* satisfiable, but any *proper* nonempty subset $S' \subsetneqq S$ is satisfiable.

(30%) **4.** A set S of *wff* is *independent* if for every wff $A \in S$, $S \setminus \{A\} \not\models A$, i.e., A is *not* implied logically by the rest of the wff in S. (So, by definition, the empty set $\emptyset$ is independent, and $S = \{A\}$ is independent iff A is *not* a tautology.)

(i) Which of the sets

(a) $\{p \Rightarrow q,\ q \Rightarrow r,\ r \Rightarrow q\}$

(b) $\{p \Rightarrow q,\ q \Rightarrow r,\ p \Rightarrow r\}$

(c) $\{p \Rightarrow r,\ r \Rightarrow q,\ q \Rightarrow p,\ r \Rightarrow (q \Rightarrow p)\}$

are independent and which are not?

(ii) Two sets of wff, S, S' are called *equivalent* if $S \models A'$ for every $A' \in S'$ and $S' \models A$ for every $A \in S$. (So, by definition, if $S = \{A\}$, where A is a tautology, $\emptyset$ is equivalent to S.) Show that for any *finite* set S of wff, there is a subset $S' \subseteq S$ which is independent and equivalent to S.