- finally
  $Q_1 y_1 \cdots Q_n y_n\ \psi(y_1, \dots, y_n, z_1, \dots, z_k)$
  equiv to
  $Q y_1' \cdots Q_n y_n'\ \psi(y_1', \dots, y_n', z_1, \dots, z_k)$
  where the y_i''s are diff from all y_j and z_ℓ.

- also, if in $Q_1 y_1 \cdots Q_n y_n$ a variable y appears more than once, can delete all instances of $Q \cdot y$ except rightmost

- using these equivalences / transformations can iteratively pull out all quantifiers
- details of proof an exercise, better to see examples ✓

Ex $(\exists x R(x,z) \wedge \neg \exists z S(z,w)) \wedge \exists x P(x,y)$

$\Leftrightarrow (\exists x R(x,z) \wedge \forall z \neg S(z,w)) \wedge \exists x P(x,y)$

$\Leftrightarrow \exists x (R(x,z) \wedge \forall z \neg S(z,w)) \wedge \exists x P(x,y)$

$\Leftrightarrow \exists x \forall z' (R(x,z) \wedge \neg S(z',w)) \wedge \exists x P(x,y)$

$\Leftrightarrow \exists x \forall z' \exists x' (R(x,z) \wedge \neg S(z',w) \wedge P(x',y))$

in p.n.f.

- Given φ in pnf, we think of consecutive $\exists$'s (and likewise consecutive $\forall$'s) as being in a single block

- we count $\forall\exists$ alternations

Def'n For φ in p.n.f, say φ is $\exists_n$ if of the form

$$\underbrace{\exists x_1^{(1)} \cdots \exists x_{k_1}^{(1)} \forall x_1^{(2)} \cdots \forall x_{k_2}^{(2)} \cdots Q x_1^{(n)} \cdots Q x_{k_n}^{(n)}}_{n\text{-blocks of quantifiers}} \; \tau$$

either $\exists$ or $\forall$ depending on parity

$\forall_n$ defined similarly (now φ begins with block of $\forall$'s)

ex: $\exists x \exists y \forall z \exists w \; \tau(x,y,z,w)$ is $\exists_3$
$\forall w \exists z \forall y \exists x \; \tau(x,y,z,w)$ is $\forall_4$

- More generally: say φ is $\exists_n/\forall_n$ if φ is logically equiv to a p.n.f. φ' which is $\exists_n/\forall_n$

~~[illegible]~~

~~[illegible]~~ note: can always increase complexity and stay logically equiv (e.g. by adding dummy quantifiers)

e.g. $\forall x \varphi(x)$ equiv to $\exists y \forall x \varphi(x)$ (since no free y's in φ)

~~[illegible]~~

First-order definability

- Fix a language L and L-structure $\mathcal{A} = \langle A, \{R_i^{\mathcal{A}}\}, \{f_j^{\mathcal{A}}\}, \{c_k^{\mathcal{A}}\} \rangle$
- there can be relations, functions, constants besides those labelled by R_i, f_j, c_k that are "implicit" in $\mathcal{A}$.

Def'n ① We say $P \subseteq A^n$ is definable in $\mathcal{A}$ if there is a formula $\varphi(x_1, \ldots, x_n)$ s.t. $\vec{a} \in P$ iff $\mathcal{A} \models \varphi(\vec{a})$

② a function $f: A^n \to A$ is definable in $\mathcal{A}$ if its graph relation $\mathrm{Graph}(f) \subseteq A^{n+1}$ is definable.

③ $a \in A$ is definable if $\{a\}$ is, i.e. if there is $\varphi(x)$ s.t. $\mathcal{A} \models \varphi(b)$ iff $b = a$.

Def'n ① $P \subseteq A^n$ is definable with parameters if there is a formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and (fixed) elements $b_1, \ldots, b_m \in A$ s.t.

$$\vec{a} \in P \text{ iff } \mathcal{A} \models \varphi(\vec{a}, b_1, \ldots, b_m)$$

② $f: A^n \to A$ is definable with parameters if $\mathrm{Graph}(f)$ is

(notice: every $b \in A$ is "definable w/ parameters" namely by fmla $\varphi(x, b) := x = b$)

<u>Ex's</u> ① Sps $\mathcal{G} = \langle G, \cdot, e\rangle$ is a group. the inversion function $f: G \to G$

$$f(x) = x^{-1}$$

is definable, by the formula

$$\varphi(x,y) := x \cdot y = e \wedge y \cdot x = e$$

(actually: "$x \cdot y = e$" suffices)

② The prime numbers $P \subseteq \mathbb{N}$ are definable in $\mathcal{N} = \langle \mathbb{N}, <, S, +, \cdot, 0 \rangle$ by

$$\varphi(x) := (\underline{1} < x) \wedge (\exists y \exists z (y \cdot z = x) \Rightarrow y = \underline{1} \vee y = x)$$

= "x is prime"

③ Consider $\mathcal{R} = \langle \mathbb{R}, +, \cdot, 0, 1 \rangle$

= "the ring of real numbers"

Consider $\varphi(x,y) := \exists z (z \neq 0 \wedge \exists w (w \cdot w = z) \wedge x + z = y)$

then $\mathcal{R} \models \varphi(a,b)$ iff $a < b$ in $\mathbb{R}$

$\Rightarrow$ $<$ is definable in $\mathcal{R}$.

<u>Note</u>: - it's a famous thm that every fmla over $\mathcal{R}$ is equiv. to a fmla w/o quantifiers

- can then show: only definable (even w/ params) subsets $A \subseteq \mathbb{R}$ in $\mathcal{R}$ are finite sets, intervals, and Boolean combos of these

$\Rightarrow$ $\mathbb{N}$ is not definable as a subset of $\mathbb{R}$ in $\mathcal{R}$!

Definability in $\mathcal{N}$

- $\mathcal{N} = \langle \mathbb{N}, <, s, +, \cdot, 0 \rangle$
- Consider simpler structure
  $\mathcal{N}_{pr} := \langle \mathbb{N}, + \rangle$
  "Presburger Arithmetic"

<u>Q</u>: How much simpler are definable sets in $\mathcal{N}_{pr}$ compared to $\mathcal{N}$?

Exercise: can define $0, s, <$ in $\mathcal{N}_{pr}$ (you try!)

It follows: definable sets in $\mathcal{N}_{pr}$ same as in $\mathcal{N}_{pr}^{+} = \langle \mathbb{N}, <, s, +, 0 \rangle$

<u>Q</u>: Can we define $\cdot$ (multiplication) in $\mathcal{N}_{pr}$?

<u>Fact</u> $A \subseteq \mathbb{N}$ is definable in $\mathcal{N}_{pr}$ iff A is <u>eventually periodic</u>
i.e. iff $\exists M, d$ s.t. $\forall n \geq M$ we have
$n \in A$ iff $n = M + kd$ for some $k \in \mathbb{N}$
(won't prove) $A = \underbrace{\cdots\cdots}_{\text{some early stuff}} \; \overset{\circ}{M} \;\; \overset{\circ}{M+d} \;\; \overset{\circ}{M+2d} \;\; \circ$

in particular, all definable sets in $\mathcal{N}_{pr}$ are recursive

$\Rightarrow$ e.g. $S = \{n^2 : n \in \mathbb{N}\}$ not definable in $\mathcal{N}_{pr}$.

Hence: $\cdot$ not definable in $\mathcal{N}_{pr}$ — if it were, S would be too:
$k \in S$ iff $\exists n (k = n \cdot n)$

(On the other hand: $+$ is definable in $\langle \mathbb{N}, S, \cdot \rangle$ ... you try)

Turns out: adding $\cdot$ to the mix increases complexity of definable sets a great deal!!

Theorem (Gödel) ① Every (partial) recursive function is definable in $\mathcal{N}$
② Every r.e. set is definable in $\mathcal{N}$.

From this we get:

Corollary. For $A \subseteq \mathbb{N}^k$, A is definable in $\mathcal{N}$ iff $A \in \bigcup \Sigma^0_n$ (i.e. iff A is arithmetical)

corollary justifies the terminology!

– Let's assume theorem for now and prove the corollary.

Pf: (sketch): $\Rightarrow$ if $A \subseteq \mathbb{N}^k$ defined by $\varphi(\vec{x})$, may assume φ in p.n.f quantifier free

- So either φ is q.f., or $\exists_n / \forall_n$ for some n.
- first check: if φ q.f. then set defined by φ (i.e. A) is recursive
  ↳ idea: induct on construction of q.f. fmlas:
  - atomic fmlas are essentially polynomial equations, i.e. if φ atomic then A is set of soln's to a poly eq'n (in k variables)
    such sets are recursive! (why?)
  - now since rec. sets closed under $\wedge$ and $\neg$, done ✓
- now follows: if $\varphi \in \exists_1$ then $A \in \Sigma^0_1$, if $\varphi \in \forall_1$ then $A \in \Pi^0_1$
- now induct: if $\exists_n \Rightarrow A \in \Sigma^0_n$
  $\forall_n \Rightarrow A \in \Pi^0_n$ ✓

($\Leftarrow$) Follows from theorem: if we can define every r.e. rel'n (Σ^0_1) w/ a fmla, then can define every Π^0_1 rel'n by taking negations
Now induct: get Σ^0_2 by adding $\exists$'s to a Π^0_1, etc.. ✓

Note: doesn't establish exact rel'n between $\exists_n$ and Σ^0_n and $\forall_n$ and Π^0_n

Now we prove Gödel's theorem.