Can define C by:
$e \in C$ iff $\forall x \exists y\, (\varphi(e,x) = y)$
"$\forall x \exists y$ (graph rel'n of r.e. → a recursive function)"
so "$\forall x \exists y\, (\Sigma_1^0)$"

Σ_1^0

Π_2^0

- So $C \in \Pi_2^0$
- does C belong to any lower complexity class? (a priori: maybe there's a def'n w/ fewer quantifiers)
- we'll show: C is Π_2^0-complete
  (follows $C \in \Pi_2^0 \setminus \Delta_2^0$, i.e. doesn't belong to a simpler class) (why? o/w could diagonalize!)

→ Fix $A \in \Pi_2^0$, $A \subseteq \mathbb{N}$
WTS: $A \leq_R C$

Know: $x \in A$ iff $\forall y\, S(x,y)$ for some Σ_1^0 S
iff $\forall y\, f(x,y)\downarrow$ for some recursive f
↑ IIZa

- Let e be code of f (i.e. $f(x,y) = \varphi(e,\langle x,y\rangle) \overset{\text{write}}{=} \varphi(e,x,y)$)
- then $x \in A$ iff $\forall y\, (\varphi(e,x,y)\downarrow)$
  iff $\forall y\, (\varphi(s(e,x),y)\downarrow)$ s-m-n function
  iff $s(e,x) \in C$

So: $x \mapsto s(e,x)$ is reduction of A to C ✓

Post's Theorem

↳ gives a connection between the arithmetic hierarchy and Turing jump.

Theorem (Post) Sps C_n is a Σ^0_n-complete set. Then:

① $A \subseteq \mathbb{N}^k$ is r.e. in C_n iff $A \in \Sigma^0_{n+1}$

② $A \subseteq \mathbb{N}^k$ is recursive in C_n iff $A \in \Delta^0_{n+1}$

C_n is a subset of $\mathbb{N}$

For $A \subseteq \mathbb{N}^k$, $A \leq_R C_n$ means $A' \leq_R C_n$ where $A' = \{\langle a_1, \dots, a_k \rangle : (a_1, \dots, a_k) \in A\}$ = encoded version of A

Pf: ② follows from ① by:

Fact: $A \in R(C_n)$ iff $A \in RE(C_n)$ and $A^c \in RE(C_n)$

Pf: relativize pf of $A \in R \Leftrightarrow A, A^c \in RE$

So we prove ①:

($\Leftarrow$) Sps $A \in \Sigma^0_{n+1}$

– then $A(\vec{x})$ iff $\exists y\, P(y, \vec{x})$ for some $P \in \Pi^0_n$

– then $P^c \in \Sigma^0_n$

$\Rightarrow P^c \leq_R C_n$ (completeness of C_n)

$\Rightarrow P^c \leq_T C_n$ ($\leq_R \Rightarrow \leq_T$)

$\Rightarrow P \leq_T C_n$ ($P \equiv_T P^c$)

$\Rightarrow P$ is rec. in C_n

Thus A is r.e. in C_n ✓

($\Rightarrow$) Sps $A \in RE(C_n)$.

Let α denote char. function of C_n

- $C_n \in \Sigma^0_n$ means the assertion "$x \in C_n$" is Σ^0_n, i.e. "$\alpha(x) = 1$" is Σ^0_n
- hence "$\neg(x \in C_n)$" is Π^0_n
  i.e. "$\alpha(x) = 0$" is Π^0_n

- since A is r.e. in C_n know by Kleene $\exists$ p.r. rel'n T and $e \in \mathbb{N}$ s.t.
  $\vec{x} \in A$ iff $\exists m\, T(e, \langle \vec{x} \rangle, \overline{\alpha}(m))$

how complex is this?

equiv to:

$$\exists n \left(\Sigma eq(n) \wedge \forall i < length(n) \left((n)_i = \alpha(i) \right) \wedge T(e, \langle \vec{x} \rangle, n) \right)$$

- everything after "$\exists n$" is p.r. except the clause "$(n)_i = \alpha(i)$"
- how complex is this?
- equiv to: $(\alpha(i) = 1 \Rightarrow (n)_i = 1) \wedge (\alpha(i) = 0 \Rightarrow (n)_i = 0)$
- equiv: $[\alpha(i) = 0 \vee (n)_i = 1] \wedge [\alpha(i) = 1 \vee (n)_i = 0]$

Σ^0_n p.r. $\Rightarrow \Sigma^0_n$; Π^0_n p.r. $\Rightarrow \Pi^0_n$

- so overall of form "$\Sigma^0_n \wedge \Pi^0_n$"
  but since $\Sigma^0_n \Pi^0_n \subseteq \Delta^0_n \subseteq \Sigma^0_{n+1}$
  also of form "$\Sigma^0_{n+1} \wedge \Sigma^0_{n+1}$" which is Σ^0_{n+1}
- hence [redacted] is "$\exists n\, \Sigma^0_{n+1}$" which is Σ^0_{n+1}
  (closure under $\exists$)

$\Rightarrow A$ is Σ^0_{n+1}

Notation: Let $0^{(n)}$ denote $0''^{\cdots'}$ ← n-times jump

Corollary: Let C_n be Σ^0_n - complete.

Then $[C_n]_T = 0^{(n)}$

doesn't matter which Σ^0_n-complete set we take

Pf: induction using prev. thm. ✓

First-order logic (FOL)

- want to study mathematical structures formally
  - sets equipped w/ functions, relations, constants

- e.g. the "structure of arithmetic"
  $\mathcal{N} := \langle \mathbb{N}, <, S, +, \cdot, 0 \rangle$
  - successor function: $S(n) = n+1$

or our favorite group
$\mathcal{G} = (G, \circ, e)$

or our fav. partial order
$\mathcal{P} = (P, \leq)$

- want to formalize:
  1. How to write statements about a structure (syntax)

② how to evaluate when a statement is true in a given structure (semantics)
③ when a statement can be formally proved.

Syntax: First-order languages

Def'n a formal language L consists of:
(i) logical symbols (these belong to every language L) i.e.:
- ctbly many variable symbols $x_1, x_2, \ldots$ (may also use y, z, etc.)
- the connectives $\wedge$ ("and") and $\neg$ ("not")
- the existential quantifier $\exists$
- equality symbol $=$
- parentheses ( ) and comma ,

(ii) its non-logical symbols (specific to a given L), i.e.:
- relation symbols $\{R_i\}$ where a given symbol R_i has arity n_i
- function symbols $\{f_j\}$ where each f_j has an arity m_j
- constant symbols $\{c_k\}$

→ to specify L, enough to specify the sets $\{R_i\}$ $\{f_j\}$ $\{c_k\}$
- write $L = \{R_i\} \cup \{f_j\} \cup \{c_k\}$

- e.g. the "language of arithmetic"
  is $L_{ar} := \{<\} \cup \{S, +, \cdot\} \cup \{0\}$
  $= \{<, S, +, \cdot, 0\}$

where:
- $<$ is a binary rel'n symbol
- S is a unary func'n symbol
- $+, \cdot$ are binary func'n symbols
- 0 is a constant symbol.

The terms of L are defined inductively:

(i) variables x_i and constant symbols c_k are terms

(ii) if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term.

- terms are syntactic objects
- but intuitively: terms represent outputs of functions obtained by composing the basic functions (represented by the symbols) f_j.

Formulas of L are also defined inductively:

"atomic formulas":

(i) if $t_1, \ldots, t_n$ are terms and R is an n-ary rel'n symbol then $R(t_1, \ldots, t_n)$ is a formula.

(ii) if s, t are terms, $s = t$ is a formula

(iii) if φ, χ are formulas, then so are $\varphi \wedge \chi$, $\neg\varphi$, $\exists x_i \varphi$.

- formulas are also syntactic
- intuitively represent assertions about terms

- we'll also use the connectives $\vee$ ("or"), $\Rightarrow$ ("implies"), $\Leftrightarrow$ ("iff") and the quantifier $\forall$
- but officially, these are abbreviations:
  $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
  $\varphi \Rightarrow \chi := \neg\varphi \vee \chi$
  $\varphi \Leftrightarrow \chi := (\varphi \Rightarrow \chi) \wedge (\chi \Rightarrow \varphi)$
  $\forall x_i \psi := \neg\exists\neg\psi$

Ex: Consider $L_{ar} = \{<, S, +, \cdot, 0\}$
- $0, x_1, x_5, S(0), +(x_1, x_5), S(\cdot(x_1, x_5))$ are all terms
- we'll abbreviate $S(0)$ as $\underline{1}$, $S(S(0))$ as $\underline{2}$, etc. (all are terms)
- will write $x_1 + x_5$ for $+(x_1, x_5)$
  $x_1 \cdot x_5$ for $\cdot(x_1, x_5)$
  etc.
- may use other abbreviations too, e.g. $x^3 + 1$ for $x \cdot (x \cdot x) + 1$
  or $\underline{2}x^3 + \underline{2}x + 1$ for $(\ldots)$, etc
- (convince yourself: all terms in this