- let $s^{-1} = t^{-1} = \emptyset$, $\ell_{-1} = 0$
- Now sps we have constructed s^{n-1}, t^{n-1} and ℓ_{n-1}

Case 1: $n = 2e+1$ is odd

subcase (i): $\exists b \in \mathbb{N}^{\mathbb{N}}$ extending t^{n-1} s.t. $\varphi^*_e(b)$ is defined (equiv: $|\varphi_e(b \restriction k)| \to \infty$)

- then define $t^n = b \restriction \ell_n$ (so $t^n \supseteq t^{n-1}$), where we choose $\ell_n > \ell_{n-1}$ long enough s.t. $|\varphi_e(t^n)| > |s^{n-1}|$ (possible by hypothesis)
- now extend s^{n-1} to s^n of length ℓ_n in any way s.t. some new entry disagrees w/ corresponding entry in $\varphi_e(t^n)$

$\Rightarrow$ thus ensuring: if α, β any extensions of s^n, t^n, will have $\varphi^*_e(\beta) \neq \alpha$

subcase (ii) $\nexists$ such b

- then let $\ell_n = \ell_{n-1} + 1$ and extend t^{n-1}, s^{n-1} to t^n, s^n of length ℓ_n arbitrarily
- for any $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ extending s^n, t^n will have $\varphi^*_e(\beta)$ undefined, hence in particular $\neq \alpha$

Case 2: $n = 2e$ is even

similar ✓

More properties of $(\mathcal{D}, \leq_T)$:

① Has least element $\underset{\sim}{0} = [\alpha]_T$ where α is any recursive oracle

② Given $\underset{\sim}{a}, \underset{\sim}{b} \in \mathcal{D}$ there is a least upper bound $\underset{\sim}{c} \geq_T \underset{\sim}{a}, \underset{\sim}{b}$, where if $\alpha \in \underset{\sim}{a}$, $\beta \in \underset{\sim}{b}$ then $\underset{\sim}{c} = [\gamma]_T$ where $\gamma(n) = \langle \alpha(n), \beta(n) \rangle$

(idea: can compute α, β from γ and if $\gamma' \geq_T \alpha, \beta$ then γ' computes γ)

③ given $\underset{\sim}{a} \in \mathcal{D}$, $\{\underset{\sim}{b} : \underset{\sim}{b} \leq_T \underset{\sim}{a}\}$ is ctbl (immediate from prev. corollary)

④ $|\mathcal{D}| = |\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}|$ "size continuum" (follows from ③ + some basic cardinal arithmetic)

⑤ For any $\underset{\sim}{a} \in \mathcal{D}$ there is $\beta \in 2^{\mathbb{N}}$ with $[\beta]_T = \underset{\sim}{a}$:

if $\underset{\sim}{a} = [\alpha]$ consider: coded graph of α

$$A = \{\langle m, n \rangle \mid \alpha(m) = n\} \subseteq \mathbb{N}$$

then ("clearly") $\chi_A \equiv_T \alpha$

↳ ⑤ gives that we can work w/ subsets of $\mathbb{N}$ (i.e. elements of $2^{\mathbb{N}}$) instead of oracles, when convenient.

A finer notion of reducibility

working w/ sets instead of oracles

Def'n Given $A, B \subseteq \mathbb{N}$, say A reduces to B (or many-to-one reduces) if there is a total recursive $f: \mathbb{N} \to \mathbb{N}$ s.t.

$$x \in A \iff f(x) \in B$$

In this case we write $A \leq_R B$

(↳ equiv: $f^{-1}(B) = A$ ↳ equiv: $\chi_A = \chi_B \circ f$)
(↳ f is called a reduction)

- idea is: if $A \leq_R B$ then to check whether $x \in A$, compute $f(x)$ and check if $f(x) \in B$
- "many-to-one" since f need not be injective.
- observe: cannot have $x \in A$, $y \notin A$ $f(x) = f(y)$
  ↳ in this sense reductions generalize characteristic functions.

Notice $A \leq_R B \Rightarrow A \leq_T B$ (immediate from $\chi_A = \chi_B \circ f$)

$\exists$ total rec. f enumerating A, hence $x \in A^c$ iff $\exists y\, (f(x) = f(y))$ r.c.

Converse not always true:
- complement A^c of A always $\equiv_T A$ (can compute χ_{A^c} from χ_A: $\chi_{A^c} = 1 - \chi_A$)
- but if A is r.c. but not recursive (e.g. A = halting problem) then if we had $A^c \leq_R A$ would have A^c r.c. (why?)
  $\Rightarrow A$ recursive, contradiction

Def'n Spse $\mathcal{C}$ is a family of subsets of $\mathbb{N}$. Say $B \subseteq \mathbb{N}$ is **$\mathcal{C}$-complete** if

1. $B \in \mathcal{C}$
2. if $A \in \mathcal{C}$ then $A \leq_R B$

Here: $RE(\alpha)$ = r.e. in α subsets of $\mathbb{N}$

Theorem For every $\alpha \in \mathbb{N}^{\mathbb{N}}$ there is $B \subseteq \mathbb{N}^{\mathbb{N}}$ that is $RE(\alpha)$-complete.
(In particular, there is an RE-complete B)

regular r.e. sets & 117a result

Pf: Let $T(e,x,u)$ be p.r. set s.t.
$A \in RE(\alpha)$ iff $\exists e \in \mathbb{N}$ s.t.
$x \in A$ iff $\exists \ell\, T(e,x,\bar{\alpha}(\ell))$

See thm on pg. 12

- Let $B = \{\langle m,n \rangle \mid \exists \ell\, T(m,n,\bar{\alpha}(\ell))\}$
- Observe B is r.e. in α (why?)

- if A is r.e. in α then $\exists e_A \in \mathbb{N}$ s.t.
$A = \{n \mid \langle e_A, n \rangle \in B\}$
i.e. $n \in A \iff \langle e_A, n \rangle \in B$
- hence $f : \mathbb{N} \to \mathbb{N}$
defined by $f(n) = \langle e_A, n \rangle$
gives reduction of A to B

$\Rightarrow B$ is $RE(\alpha)$-complete ✓

$\hookrightarrow$ We can use the theorem to define a canonical degree above a given $\mathcal{L} \in \mathcal{D}$

<u>Def'n</u> Given a Turing degree $\underset{\sim}{a} = [\alpha]_T$ define $\underset{\sim}{a}'$ (the <u>jump</u> of $\underset{\sim}{a}$) to be $[B]_T$, where B is any RE(α)-complete set.

↳ note: if B and B' are both RE(α)-complete then $B \equiv_R B'$, hence $B \equiv_T B'$ and $[B]_T = [B']_T$, so def'n makes sense.

⟿ generalization of "∃ r.e. set that is not rec." pf is the same

<u>Claim</u> $\underset{\sim}{a} <_T \underset{\sim}{a}'$

<u>Pf</u>: - Let B be RE(α)-complete.
- suffices to check $B \not\leq_T \alpha$
- if $B \leq_T \alpha$, then since B is RE(α)-complete every RE(α)-rel'n is $\leq_T \alpha$ (i.e. recursive in α)
- in particular, the universal rel'n
$W(e,x)$ iff $\exists k T(e, x, \bar{\alpha}(k))$
is recursive in α
$\Rightarrow P(n)$ iff $W(n,n)$ is $R(\alpha)$ too
$\Rightarrow \neg P(n)$ is too (rec rel'ns closed under $\neg$)
$\Rightarrow$ for some e

closed under tot rec. substitution

$\neg P(n)$ iff $\exists k T(e, n, \bar{\alpha}(k))$

but then
$\neg P(e)$ iff $\exists k T(e, e, \bar{\alpha}(k))$ ← by above
iff $\neg \exists k T(e, e, \bar{\alpha}(k))$ ← by def'n

contradiction.