- in this sense T is at least as strong as Q_0 despite being written in a different language.
- Then we have:

<u>Thm</u> (1st Incompleteness theorem, general form)

Sps T is a consistent, recursive collection of sentences in a language L and there is an interpretation of Q_0 in T s.t. the map $\langle \varphi \rangle \to \langle \varphi^* \rangle$ is recursive.

<u>Then</u>: T is undecidable and incomplete.

<u>Gödel's approach</u>: an explicit independent sentence.

<u>Idea</u>: find σ that asserts "σ is not provable"

<u>More explicitly</u>: want a fmla that asserts "the fmla w/ Gödel code no. is not provable" ... whose Gödel code is no.!

↳ Sps $k(n)$ is a rec. function that on input n returns the Gödel code of the fmla asserting "the fmla w/ Gödel code n is not provable"

↳ want a lemma that gives an n s.t. $k(n) = n$

- Sps $L \supseteq \{0, s\}$ and Q is a collection of L-sentences that represents every rec. f/R

<u>Lemma</u> (Gödel fixed pt lemma)

For every L-fmla $\varphi(x)$ w/ one free variable, there is a sentence σ s.t.

$$Q \vdash \sigma \Leftrightarrow \varphi(\underline{\langle \sigma \rangle})$$

(Imagine then: $\varphi(n)$ asserts "fmla coded by n is not provable")

<u>Pf</u>: let

$$Sub(n,m) = \begin{cases} \langle \Phi(\underline{m}) \rangle & \text{if } n \text{ codes a fmla } \Phi(x) \\ 0 & \text{o/w} \end{cases}$$

- as before, take for granted $Sub(n,m)$ is recursive
- let $\chi(x,y,z)$ be fmla representing $Sub(x,y) = z$ in Q

- let $\psi(x)$ be the fmla:
$$\exists z\,(\chi(x,x,z) \wedge \varphi(z))$$
(read as "$\varphi(Sub(x,x))$")

- let $n_0 = \langle \psi(x) \rangle$
- we claim $\sigma := \psi(\underline{n_0})$ works
$(= \psi(\underline{\langle \psi(x) \rangle}))$

<u>Observe</u> $\langle \sigma \rangle = \langle \psi(\underline{n_0}) \rangle = Sub(n_0, n_0)$

☆ so: $Q \vdash \varphi(\underline{\langle \sigma \rangle}) \Leftrightarrow \varphi(\underline{Sub(n_0,n_0)})$
(a triviality ... these are literally the same fmla ... just setting up here)

- Write m for $Sub(n_0, n_0)$
- Since χ reps sub in Q, have:
$$Q \vdash \chi(\underline{n_0}, \underline{n_0}, \underline{m})$$

- Hence $Q \vdash \varphi(\underline{m}) \Leftrightarrow (\chi(\underline{n_0},\underline{n_0},\underline{m}) \wedge \varphi(\underline{m}))$
(trivial again, since $Q \vdash \chi(\underline{n_0},\underline{n_0},\underline{m})$)

- Hence $Q \vdash \varphi(\underline{m}) \Leftrightarrow \exists z\,(\chi(\underline{n_0},\underline{n_0},z) \wedge \varphi(z))$

since $Q \vdash$ uniqueness of outputs of χ

"the output of $Sub(n_0,n_0)$ is m and m satisfies φ"

"the output of $Sub(n_0,n_0)$ satisfies φ"

i.e. $Q \vdash \varphi(\underline{Sub(n_0, n_0)}) \Leftrightarrow \exists z\, (K(\underline{n_0}, \underline{n_0}, z) \wedge \varphi(z))$

but this is σ.

i.e. $Q \vdash \varphi(\underline{\langle\sigma\rangle}) \Leftrightarrow \sigma$ ✓

Define as before a relation
$Proof_Q(x, y)$
iff y codes a proof of x from Q, i.e.
$Proof_Q(y) \wedge (y)_{length(y)-1} = x$

- then $Proof_Q(x,y)$ is recursive, hence rep'able in Q, say by a fmla that we also (abusing notation) denote $Proof_Q(x,y)$

- Define a relation $Prvbl_Q(x)$ as: $\exists y\, Proof_Q(x,y)$

- let $\varphi(x)$ be the fmla $\neg Prvbl_Q(x)$
- if we apply fixed pt lemma to this we get a sentence σ_Q s.t.

(**) $Q \vdash \sigma_Q \Leftrightarrow \neg Prvbl(\underline{\langle\sigma_Q\rangle})$

- We call σ_Q the Gödel sentence of Q.
- WTS: σ_Q is independent from Q, i.e. $Q \nvdash \sigma_Q$ and $Q \nvdash \neg\sigma_Q$
- turns out: need to make a technical assumption about Q for this to be possible.

<u>We assume</u>: Q is <u>ω-consistent</u>, i.e. for each fmla $\psi(y)$, if $Q \vdash \exists y\, \psi(y)$ then there is $p \in \mathbb{N}$ $Q \nvdash \neg\psi(\underline{p})$

(i.e. if Q proves there is a witness to ψ, it doesn't prove there is no standard witness)

<u>Fact</u>: Robinson arithmetic, Peano, etc. are ω-consistent (... in ZFC)

<u>Notice</u>: if ψ represents a relation R then $Q \nvdash \neg\psi(\underline{p})$ implies $Q \vdash \psi(\underline{p})$

Here is Gödel's original formulation

<u>Theorem</u> (1st incomp. thm, o.g. form)
Assume Q is recursive, ω-consistent, and represents every recursive R/F then $Q \nvdash \sigma_Q$ and $Q \nvdash \neg\sigma_Q$

<u>Pf</u>. Sps first $Q \vdash \sigma_Q$
- then there is $k \in \mathbb{N}$ s.t. $\mathrm{Proof}_Q(\langle\sigma_Q\rangle, k)$
- by representability of Proof_Q, get $Q \vdash \mathrm{Proof}_Q(\underline{\langle\sigma_Q\rangle}, \underline{k})$
- Hence $Q \vdash \exists y\, \mathrm{Proof}_Q(\underline{\langle\sigma_Q\rangle}, y)$
  i.e. $Q \vdash \mathrm{Prvbl}_Q(\underline{\langle\sigma_Q\rangle})$
- OTOH get (from $Q \vdash \sigma_Q$) $Q \vdash \neg\mathrm{Prvbl}_Q(\underline{\langle\sigma_Q\rangle})$ from (**), contradicting consistency of Q

- Now sps $Q \vdash \neg\sigma_Q$
- then by a deduction from (**)
  $Q \vdash Prvbl_Q(\underline{<\sigma_Q>})$
  i.e. $Q \vdash \exists y\, Proof_Q(\underline{<\sigma_Q>}, y)$

- By ω-consistency of Q and rep'bility of $Proof_Q$ there is $p \in \mathbb{N}$ s.t.
  $Q \vdash Proof_Q(\underline{<\sigma_Q>}, \underline{p})$

- But $Proof_Q$ here (i.e. the fmla) represents the real $Proof_Q$ relation on $\mathbb{N}$
- Hence $Proof_Q(<\sigma_Q>, p)$ really holds i.e. p codes an actual proof of σ_Q i.e. $Q \vdash \sigma_Q$, contradiction ✓

Rosser's trick

- it is possible, if we make slightly stronger assumptions about Q ($Q_0 \subseteq Q$ enough), to dispense w/ ω-consistency hypothesis
- Rosser did this: showed, for consistent and recursive such Q, there is a fmla ρ_Q s.t.
  $Q \nvdash \rho_Q$ and $Q \nvdash \neg\rho_Q$

- we omit the pf.