Pf of Tarski:

- sps $\langle Th(\mathcal{N})\rangle$ is arithmetical
- Define $\mathcal{U}$ by

$(e,m) \in \mathcal{U}$ iff

"$e = \langle \varphi(x)\rangle$ for some fmla $\varphi(x)$ w/ one free var. and $\mathcal{N} \models \varphi(\underline{m})$"

- define a (partial) function $Sub: \mathbb{N}^2 \to \mathbb{N}$ by $Sub(e,m) = k$ iff $e = \langle \varphi(x)\rangle$ is a code for a fmla $\varphi(x)$ and $k = \langle \varphi(\underline{m})\rangle$
- then Sub is "clearly" recursive
- now we can write defn of $\mathcal{U}$ as $\exists k\,(Sub(e,m) = k \wedge k \in \langle Th(\mathcal{N})\rangle)$

- makes clear $\mathcal{U}$ is arithmetical, since $\langle Th(\mathcal{N})\rangle$ is.
- we claim $\mathcal{U}$ is universal, which gives a contradiction

- If $B \subseteq \mathbb{N}$ is arithmetical then (as we showed) there is a fmla defining B, i.e. some $\varphi(x)$ s.t.

$m \in B$ iff $\mathcal{N} \models \varphi(m)$

- but then if $e = \langle \varphi\rangle$ we have

$(e,m) \in \mathcal{U}$ iff $\varphi(\underline{m}) \in Th(\mathcal{N})$

iff $\mathcal{N} \models \varphi(\underline{m})$

iff $\mathcal{N} \models \varphi(m)$

$\Rightarrow \mathcal{U}$ is universal ✓

- From Tarski's theorem we can deduce a weak form of Gödel's 1st incompleteness theorem.

<u>Theorem</u> (1st Incomp. Thm.; weak form)
Sps Σ is a collection of formulas (in lang. of arithmetic)
If $\mathcal{N} \models \Sigma$ (so that in particular Σ is consistent) and Σ is recursive, then Σ is not complete.
In particular, there is no recursive axiomatization of $Th(\mathcal{N})$.

<u>Pf</u>: - We showed $\Sigma \vdash \varphi \Rightarrow \Sigma \models \varphi$
- Hence $Conseq(\Sigma) \subseteq Th(\mathcal{N})$
- if Σ is complete then $Conseq(\Sigma) = Th(\mathcal{N})$
  <u>why</u>: for each $\varphi \in Th(\mathcal{N})$ either $\Sigma \vdash \varphi$ or $\Sigma \vdash \neg\varphi$ by completeness
  hence must prove φ
  since $\neg\varphi \notin Th(\mathcal{N})$ (φ is true in $\mathcal{N}$!)
- By prev. prop'n, $Conseq(\Sigma)$ is r.e. (and in fact recursive, since Σ complete)
- but $Th(\mathcal{N})$ is not even arithmetical
  $\Rightarrow \Sigma$ is incomplete ✓

↳ Thus for any recursive $\Sigma \subseteq Th(\mathcal{N})$ there are fmlas $\varphi \in Th(\mathcal{N})$ s.t. $\Sigma \nvdash \varphi$
must also have $\Sigma \nvdash \neg\varphi$ — say: φ is <u>independent</u> from Σ

Toward Gödel's Theorems

- Gödel's 1st Incomp. Thm. stronger than the one above
- not based on notion of <u>definability</u>, which is a semantic concept (depends on whether $\mathcal{N} \models \varphi(m)$ and hence on the underlying structure $\mathcal{N}$)
- ... but rather <u>representability</u> (defined below), which is syntactic and doesn't refer to specific structure in our language (such as $\mathcal{N}$)

- Fix a lang. L containing the constant symbol 0 and (unary) function symbol s (think: L = lang. of arithmetic, or perhaps some extension)
- thus for each $m \in \mathbb{N}$:

$$\underline{m} := \underbrace{s(s(\cdots(s(0))\cdots))}_{m\text{-times}}$$

is an L-term

- Let Q be any fixed collection of ~~[illegible]~~ L-sentences

- <u>Say</u>: a rel'n $R \subseteq \mathbb{N}^n$ is <u>representable in Q</u> if there is a fmla φ s.t. for each $x_1, \ldots, x_n \in \mathbb{N}$ we have:

$$R(x_1,\dots,x_n) \Rightarrow Q \vdash \varphi(\underline{x_1},\dots,\underline{x_n})$$
and $$\neg R(x_1,\dots,x_n) \Rightarrow Q \vdash \neg\varphi(\underline{x_1},\dots,\underline{x_n})$$

Also say: a (partial) function $f: \mathbb{N}^n \to \mathbb{N}$ is <u>representable</u> <u>in</u> <u>Q</u> if there is $\varphi(x_1,\dots,x_n)$ s.t.

$$Q \vdash \forall x_1 \ldots x_n, y, z\ [\varphi(x_1,\dots,x_n,y) \wedge \varphi(x_1\dots x_n, z) \Rightarrow y = z]$$

<u>and</u> if $f(x_1,\dots,x_n) = y$, then
$$Q \vdash \varphi(\underline{x_1},\dots,\underline{x_n},\underline{y})$$

<u>Observe</u>: if $\mathcal{N} \models Q$ (think: Q an attempt to axiomatize $Th(\mathcal{N})$) then if R is representable in Q, R is definable in $\mathcal{N}$ since
$$Q \vdash \varphi(\vec{\underline{x}}) \Rightarrow \mathcal{N} \models \varphi(\vec{x})$$
$$Q \vdash \neg\varphi(\vec{\underline{x}}) \Rightarrow \mathcal{N} \models \neg\varphi(\vec{x})$$
so φ defines R (over $\mathcal{N}$)

likewise if Q represents f then f is definable in $\mathcal{N}$

<u>But</u>: representability stronger than definability in general: if $R \subseteq \mathbb{N}^n$ rep'able over Q by φ, for every $\vec{x} \in \mathbb{N}^n$ can witness whether $\vec{x} \in R$ or $\vec{x} \notin R$ w/ a <u>proof</u> from Q.

(follows: if Q is recursive and R rep'able in Q, then R is recursive ... why?)

<u>Theorem</u> Sps L contains the symbols $0, s$ and Q is a consistent set of sentences in this language.

If every recursive relation and function is representable in Q, then $\langle \mathrm{Conseq}(Q) \rangle$ is <u>not</u> representable in Q.

($\mathrm{Conseq}(Q)$ here is analogue of $\mathrm{Th}(\mathcal{N})$ in Tarski's theorem: $\mathcal{N} \models \varphi$ repl. by $Q \vdash \varphi$)

<u>Pf.</u> (Sketch) Similar to Tarski

- first check that Q-rep'able rel'ns are closed under complements and total recursive substitutions

(if $\varphi(\vec{x})$ reps $R(\vec{x})$ then $\neg\varphi(\vec{x})$ reps R^c; if $\psi(\vec{x}, y)$ reps $f(\vec{x}) = y$ for f total recursive, then $\exists k(\psi(\vec{x}, k) \wedge \varphi(\vec{x}, k))$ reps $R(\vec{x}, f(\vec{x}))$ ... can you find the formal proofs?)

$\Rightarrow$ implies, by a diagonal argument, there is no universal Q-rep'able set, i.e. no Q-rep'able $U \subseteq \mathbb{N}^2$ s.t. for every Q-rep'able $B \subseteq \mathbb{N}$ there is $e \in \mathbb{N}$ s.t. $m \in B \Leftrightarrow (e, m) \in U$.

$\hookrightarrow$ if there were, could find $\varphi(x)$ (rep'ing the rel'n $\neg U(n, n)$) s.t. $Q \vdash \varphi(\underline{n})$ and $Q \vdash \neg\varphi(\underline{n})$ for some n, contradicting consistency.

- details left to you ...

- now sps $\langle Conseq(Q) \rangle$ is Q-rep'able, say by $\psi(x)$
- we'll show there is a universal Q-rep'able set, contradiction.

- define: $U(e,m)$ iff e codes a fmla $\varphi(x)$ and $Q \vdash \varphi(\underline{m})$

  i.e. iff $\exists k\, (Sub(e,m) = k \wedge k \in \langle Conseq(Q) \rangle)$

- By Tarski-like arg, clear that U is universal. We claim it is Q-rep'able.
- By hypothesis $sub(e,m) = k$ is rep'able (since recursive) say by φ
- Then: $\tau(e,m) := \exists k\, (\varphi(e,m,k) \wedge \psi(k))$ represents U

  __why__ (briefly): wts:

  $U(e,m) \Rightarrow Q \vdash \tau(\underline{e},\underline{m})$

  $\neg U(e,m) \Rightarrow Q \vdash \neg\tau(\underline{e},\underline{m})$

  e.g. if $U(e,m)$, then if $k = Sub(e,m)$ have $Q \vdash \varphi(\underline{e},\underline{m},\underline{k})$ and $Q \vdash \psi(\underline{k})$

  since Q reps φ ; since $k \in \langle Conseq(Q) \rangle$

  hence $Q \vdash \varphi(\underline{e},\underline{m},\underline{k}) \wedge \psi(\underline{k})$

  hence $Q \vdash \exists k\, (\varphi(\underline{e},\underline{m},k) \wedge \psi(k))$

  ... assuming our deduction system can make these moves (it can)

Follows from theorem: if Q reps all recursive rel'ns/func'ns then $\langle \mathrm{Conseq}(Q) \rangle$ is not recursive.

So now we ask: how rich need Q be to represent all rec. R's and F's?
↪ we'll show there is a recursive (in fact finite!) such Q.

Def'n Robinson's Arithmetic Q. is the following collection of axioms (in the language of arithmetic $L = \{<, s, +, \cdot, 0\}$)

1. $\forall x\,(s(x) \neq 0)$
2. $\forall x\,(x \neq 0 \Rightarrow \exists y\,(s(y) = x))$
3. $\forall x \forall y\,(s(x) = s(y) \Rightarrow x = y)$
4. " < is a linear ordering"
   (i.e. "< is irreflexive, transitive, total")
5. $\forall x \forall y\,(x < s(y) \Rightarrow x = y \vee x < y)$
6. $\forall x\,(0 \leq x)$
7. $\forall x\,(x + 0 = x) \wedge \forall x \forall y\,(x + s(y) = s(x+y))$
8. $\forall x\,(x \cdot 0 = 0) \wedge \forall x \forall y\,(x \cdot s(y) = x \cdot y + x$

If $\chi_R(\vec{x}) = \mathcal{E}$ rep'd by $e(\vec{x}, \varepsilon)$ then R rep'd by $e(\vec{x}, 1)$

Theorem every recursive function (and hence relation) is representable in Q.