Ma 117b Homework #5

Due Tuesday, March 11th at 1:00pm

1. (40 points) Let Q_0 denote Robinson's Arithmetic. Characterize the relations $R \subseteq \mathbb{N}^n$ that are representable in Q_0.

   Call a relation $R \subseteq \mathbb{N}^n$ *weakly representable* in Q_0 if there is a formula $\varphi(x_1, \dots, x_n)$ such that for all $k_1, \dots, k_n \in \mathbb{N}$ we have
   $$R(k_1, \dots, k_n) \Leftrightarrow Q_0 \vdash \varphi(\underline{k}_1, \dots, \underline{k}_n).$$
   Characterize the relations that are weakly representable in Q_0.

2. (30 points) Let L be a finite first-order language. A *theory* in L is any set of sentences T closed under provability (i.e., if $T \vdash \sigma$, then $\sigma \in T$, for any sentence σ; i.e. $T = \text{Conseq}(T)$).

   We call a theory T *recursively axiomatizable* if there is a recursive set of sentences $S \subseteq T$ such that $T = \text{Conseq}(S) = \{\sigma : S \vdash \sigma\}$. Show that T is recursively axiomatizable iff it is r.e. (i.e., the set of codes $\langle T \rangle = \{\langle \sigma \rangle : \sigma \in T\}$ is r.e.).

3. (30 points) Let L be a finite first order language containing the symbols $0, S$. Let Q be a set of sentences in this language such that the recursive relations and functions are representable in Q. A formula $\varphi(x)$ is called a *truth definition* in Q if for every sentence σ of L we have
   $$Q \vdash \sigma \Leftrightarrow \varphi(\underline{\langle \sigma \rangle}).$$
   Show that if Q is consistent, there is no such truth definition.

4. (40 points) Consider the language of arithmetic and let Q be any consistent, recursive set of sentences in this language containing Q_0. View every formula in this language as a word in a finite alphabet (as in class) and define for each formula φ its *length* to be $L(\varphi) =$ the number of symbols in φ. If $u = (\varphi_1, \dots, \varphi_n)$ is a finite sequence of formulas, put $L(u) = L(\varphi_1) + \cdots + L(\varphi_n)$. If $Q \vdash \varphi$, let
   $$|\varphi|_Q = \min\{L(u) : u \text{ is a proof of } \varphi \text{ from } Q\}.$$
   Let also $|\varphi|_Q = \infty$, if φ is not provable from Q. Note that:

   1) $Q \subseteq Q' \Rightarrow |\varphi|_Q \geq |\varphi|_{Q'}$,

   2) The relation
   $$P(\langle \varphi \rangle, n) \Leftrightarrow |\varphi|_Q \leq n$$
   is recursive,

   3) $|\varphi|_{Q \cup \{\sigma\}} \leq |\sigma \Rightarrow \varphi|_Q + L(\sigma) + L(\varphi)$.

   Prove the following:

   (i) For any recursive total function $g : \mathbb{N} \to \mathbb{N}$, there are infinitely many sentences σ such that $Q \vdash \sigma$ and $|\sigma|_Q > g(L(\sigma))$.

   (ii) Suppose that neither $Q \vdash \sigma_0$ nor $Q \vdash \neg\sigma_0$. Show that for every recursive total function $g : \mathbb{N} \to \mathbb{N}$, there are infinitely many sentences σ such that $Q \vdash \sigma$ and $|\sigma|_Q > g(|\sigma|_{Q \cup \{\sigma_0\}})$.