Ma 117b Homework #4

Due Tuesday, February 25th at 1:00pm

1. (40 points) It follows from the existence of universal r.e. sets and the Matiyasevich Representation Theorem (MRTh) that there is a universal diophantine equation (for any given number of parameters). For one parameter, this means that there is a diophantine equation

   $$U(a, n, x_1, \dots, x_m) = 0$$

   with two parameters a, n such that every diophantine set $D \subseteq \mathbb{N}$ can be written as

   $$a \in D \Leftrightarrow \exists x_1 \dots \exists x_m U(a, n, x_1, \dots, x_m) = 0,$$

   for some n.

   Prove this directly without assuming the existence of universal r.e. sets and the (MRTh). You can use however that the diophantine relations are closed under $\wedge, \vee$, existential quantification, bounded universal quantification and substitution by diophantine functions.

   Hint. Make use of the Cantor pairing function $J(x, y)$ (see previous homework) and its inverses K, L, where $x = J(K(x), L(x))$, to code up polynomials with positive integer coefficients.

2. (40 points) Show that in general there is no computable bound on the size of a solution of a diophantine equation, when such a solution exists. More precisely, show that there is a diophantine equation $F(a, x_1, \dots, x_n) = 0$, where F is a polynomial with integer coefficients, with parameter a and unknowns $x_1, \dots, x_n$, such that for any total computable function $B : \mathbb{N} \to \mathbb{N}$, there is $a \in \mathbb{N}$ such that $F(a, x_1, \dots, x_n) = 0$ has a solution in $\mathbb{N}$ but it has no solution in $\mathbb{N}$ with $x_1, \dots, x_n \leq B(a)$. (You may assume the (MRTh).)