Ma 117b Homework #3

Due Thursday, February 13th at 1:00pm

1. (15 points) Find formulas in prenex normal form equivalent to the following:

   i) $\neg\exists x[\forall z P(x,y,z) \Rightarrow \neg\exists y(Q(y) \wedge R(z))]$,

   ii) $\neg\forall x\forall y[(R(z) \Rightarrow \neg\forall z\forall w(P(x,y,z) \vee \exists v\neg Q(v,w))]$.

2. (25 points) Consider the structure of arithmetic $\mathcal{N} = \langle \mathbb{N}, <, +, \cdot, S, 0\rangle$. Using the Matiyasevich Representation Theorem, show that for any $n \geq 1$ a relation $R \subseteq \mathbb{N}^k$ is Σ^0_n (resp., Π^0_n) iff it is definable by a $\exists_n$ (resp., $\forall_n$) formula in the language of arithmetic.

3. (25 points) Call a partial function $f : \mathbb{N}^k \to \mathbb{N}$ *min-recursive* iff it is built up from the following functions:

   (i) $x \mapsto 0$,

   (ii) $x \mapsto x+1$,

   (iii) $+, \cdot$,

   (iv) $(x_1, \dots, x_n) \mapsto x_i$, for $1 \leq n, 1 \leq i \leq n$,

   (v) The characteristic function $\chi_=$ of equality,

   using composition and minimalization (but no primitive recursion). Show that a function if min-recursive iff it is recursive.

4. (25 points) Show (using the solution to the Hilbert Tenth Problem) that there is no algorithm to test whether an arbitrary polynomial (in several variables) with positive integer coefficients takes a square value (for some nonnegative argument).

   Hint. Recall the Cantor pairing function

   $$J(x,y) = \frac{(x+y)(x+y+1)}{2} + x.$$

   When is $2J(x,y)+1$ a square?

5. (25 points) Let $F(x_1, \dots, x_n)$ be a non-constant polynomial with integer coefficients. Prove that for some non-negative integers $a_1, \dots, a_n$, the value $F(a_1, \dots, a_n)$ is non-zero and composite (positive or negative). (Thus any prime representing polynomial takes a negative composite value for some non-negative argument.)