Ma 117b Homework #2

Due Tuesday, February 4th at 1:00pm

1. (20 points) Prove the following facts from the notes. For every $n \in \mathbb{N}$ we have:
   i) Δ^0_n does not have the enumeration property,
   ii) Σ^0_n and Π^0_n are not closed under complements,
   iii) Σ^0_n is not closed under $\forall$ quantification, and Π^0_n is not closed under $\exists$ quantification,
   iv) $\Sigma^0_n \cup \Pi^0_n \subsetneq \Delta^0_{n+1}$ and $\Delta^0_n \subsetneq \Sigma^0_n, \Pi^0_n$.

2. (30 points) Let $\{\varphi_e\}$ be an acceptable, effective enumeration of the partial recursive functions. Show that the set $\{e : \varphi_e \equiv 0\}$ is complete Π^0_2.

3. (15 points) Consider the structure of arithmetic $\mathcal{N} = \langle \mathbb{N}, <, +, \cdot, S, 0 \rangle$
   a) Translate into the language of arithmetic the following sentences or expressions in English:
      (i) Every nonzero natural number is the successor of some natural number.
      (ii) x is divisible by y.
      (iii) 5 is a prime number.
      (iv) Every natural number is the sum of four squares (of natural numbers).
      (v) The product of any two consecutive natural numbers is even.
      (vi) x and y are prime to each other.
      (vii) For all x, y, z if x divides $y \cdot z$ and x is prime to y, then x divides z.
   b) Translate into English the following:
      (i) $\neg \forall x \exists y (y^2 = x)$
      (ii) $\forall x [\exists y (x = 2 \cdot y) \Rightarrow \exists y (x^2 = 4 \cdot y)]$
      (iii) $\forall x \exists y [x < y \wedge y \neq 1 \wedge \neg \exists z \exists w (z \cdot w = y \wedge z < y \wedge w < y)]$
      (iv) $\forall x \forall y [x < y \Rightarrow \exists z (z \neq 0 \wedge x + z = y)]$
      (v) $\forall x \forall y [(\exists z (x = 2 \cdot z) \wedge \exists w (y = 2w)) \Rightarrow \exists u (x + y = 2 \cdot u)]$

4. (15 points) Find the free and bound variables in the formulas below:
   i) $\forall x \exists y P(y, x)$
   ii) $\forall y [R(x, y) \Rightarrow \exists x Q(x)]$
   iii) $R(x, a) \vee R(y, b)$
   iv) $\forall x R(x, y) \vee \exists y [R(x, y) \wedge Q(y)]$

5. (15 points) In which of the following formulas is there a free occurrence of x?
   i) $\forall x R(x)$,
   ii) $\exists y \forall x P(y, z)$,
   iii) $\forall x [P(x, y) \Rightarrow \forall y Q(x)]$,
   ii) $\forall x \exists y P(x, y) \wedge \forall y (Q(y) \Rightarrow P(x, x))$

6. (25 points)
   a) Prove the following formulas are not logically valid:
      i) $\forall x (P(x) \vee Q(x)) \Rightarrow (\forall x P(x) \vee \forall x Q(x))$
      ii) $(\exists x P(x) \wedge \exists x Q(x)) \Rightarrow \exists x (P(x) \wedge Q(x))$
      iii) $(\forall x P(x) \Rightarrow \forall x Q(x)) \Rightarrow \forall x (P(x) \Rightarrow Q(x))$

iv) $(P(x) \Leftrightarrow Q(x)) \Leftrightarrow (x = y)$

b) Does the sentence $\forall x \forall y \forall z[(R(x,y) \wedge R(y,z)) \Rightarrow R(x,z)]$ logically imply the sentence $\forall x \forall y[R(x,y) \vee R(y,x)]$?

7. (15 points)

a) Consider the first order language $L_0 = \emptyset$ with no nonlogical symbols. Its structures consist only of nonempty sets A, i.e., they have the form $\mathcal{A} = \langle A \rangle$.

i) If $\langle A \rangle \models \forall x \forall y(x = y)$, what can you say about A?

ii) If $\langle A \rangle \models \exists x \exists y(x \neq y)$, what can you say about A?

iii) If $\langle A \rangle \models \exists x \exists y \exists z[x \neq y \wedge y \neq z \wedge x \neq z \wedge \forall w(w = x \vee w = y \vee w = z)]$, what can you say about A?

b) Consider the first order language whose only nonlogical symbol is the binary relation symbol R. Its structures are of the form $\mathcal{A} = \langle A, R^{\mathcal{A}} \rangle$, where $R^{\mathcal{A}}$ is a binary relation on A. For simplicity, we will just write $\langle A, R \rangle$.

i) Write a sentence σ of this language such that $\langle A, R \rangle \models \sigma$ iff $R = A \times A$.

ii) Write a sentence τ of this language such that $\langle A, R \rangle \models \tau$ iff $R = \emptyset$.

iii) Write a sentence π of this language such that $\langle A, R \rangle \models \pi$ iff R contains exactly one pair.