Ma 117b Homework #1

Due Tuesday, January 21st at 1:00pm

1. (70 points) A non-negative real number r is *recursive* if there are total recursive functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $g(n) \neq 0$, if $n \neq 0$, and for $n \neq 0$, $|r - \frac{f(n)}{g(n)}| < \frac{1}{n}$. A negative real number r is recursive if $-r$ is recursive.

   a. Show that a real number r is recursive iff the set of rationals $< |r|$ is recursive in the sense that the relation
   $$P_r(m,n) \Leftrightarrow n \neq 0 \;\&\; \frac{m}{n} < |r|$$
   is recursive.

   b. Show that a real r is recursive if $|r|$ has a recursive decimal expansion, i.e., there is total recursive $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \neq 0, f(n) \leq 9$ and $|r| = \sum_{n=0}^{\infty} f(n)10^{-n}$.

   c. Show that the recursive reals form a field, i.e., they are closed under addition, subtraction, multiplication and division (by non-zero numbers).

   d. Show that e, π are recursive.

   e. Show that the real solutions of a one variable polynomial with recursive coefficients are recursive. *Hint:* Use Sturm's algorithm. If you don't know what that is look it up in an algebra book, like van der Waerden's book.

2. (30 points) Fill in the primitive recursion part of the proof we left out in class. That is, show that if $\alpha : \mathbb{N} \to \mathbb{N}$ is an oracle, and g, h are (partial) functions defined by primitive recursive relations P_g and P_h in the sense that
   $$\begin{aligned} g(x) = y &\quad\Leftrightarrow\quad \exists k\, P_g(x, y, \overline{\alpha}(k)) \\ h(a,b,c) = y &\quad\Leftrightarrow\quad \exists k\, P_h(a, b, c, y, \overline{\alpha}(k)) \end{aligned}$$
   then if f is defined recursively by
   $$\begin{aligned} f(0,x) &= g(x) \\ f(n+1,x) &= h(f(n,x), n, x) \end{aligned}$$
   then there is a primitive recursive relation P_f such that
   $$f(n,x) = y \quad\Leftrightarrow\quad \exists k\, P_f(n, x, y, \overline{\alpha}(k)).$$