Ma/CS 117a HW #1

Due Tuesday, October 15th, at 1pm

1) Let $H : \mathbb{N} \to \mathbb{N}$ be defined by:

$$H(n) = \begin{cases} 0 & \text{if there is a sequence of at least } n \text{ consecutive 7s in the decimal expansion of } \pi, \\ 1 & \text{otherwise.} \end{cases}$$

Is $H(n)$ primitive recursive? Justify your answer.

2) Let $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ denote the basis of the natural logarithm. Define a function $g : \mathbb{N} \to \mathbb{N}$ by

$$g(n) = \lfloor ne \rfloor = \text{the largest integer } \leq ne.$$

Show that g is primitive recursive. Use this to show that the function $f : \mathbb{N} \to \mathbb{N}$ defined be $f(n) = a_n$, where a_n denotes the nth digit in the decimal expansion of e, is primitive recursive.

Hint: Show that there is no integer x such that

$$n \sum_{k=0}^{n} \frac{1}{k!} < x < n \left(\frac{1}{n!} + \sum_{k=0}^{n} \frac{1}{k!} \right).$$

Use this to show $\lfloor ne \rfloor = \lfloor \sum_{k=0}^{n} \frac{n}{k!} \rfloor$.

3) Consider the following type of recursion (a form of *recursion with substitution in the parameters*):

$$\begin{aligned} f(0, n) &= g(n) \\ f(m+1, n) &= h(f(m, S(m, n))) \end{aligned}$$

where h, g, S are given functions on $\mathbb{N}$. Show that if h, g, S are primitive recursive, then so is f.

Hint: Notice that for $m > 0$, we have

$$\begin{aligned} f(m, n) &= h(f(m-1, S(m-1, n))) \\ &= h^{(2)}(f(m-2, S(m-2, S(m-1, n)))) \\ &\vdots \\ &= h^{(m)}(g(S(0, S(1, S(2, \ldots, S(m-2, S(m-1, n)) \ldots))))) \end{aligned}$$

where $h^{(k)}$ denotes h followed by $k-1$ parentheses.

Look at the numbers

$$n, S(m-1, n), S(m-2, S(m-1, n)), \ldots$$

which occur in building up this expression.

4) Consider the following type of recursion on $\mathbb{N}$, which is a special case of a form of recursion called *unnested double recursion*:

$$\begin{aligned} f(0, n) &= g(n) \\ f(m+1, 0) &= h(m) \\ f(m+1, n+1) &= p(f(m+1, n), f(m, n+1), m, n) \end{aligned}$$

Show that if the functions g, h, and p are primitive recursive, then so is f.

Hint: Define the auxiliary function

$$F(k) = \langle f(0, k), f(1, k-1), \ldots, f(k, 0) \rangle.$$

Express f in terms of F and show that F is primitive recursive.

5) Let $\sqrt{2} = a_0.a_1a_2\ldots$ be the decimal expansion of the square root of 2 (so $a_0 = 1, a_1 = 4, a_2 = 1$, etc.). Show that the function $f : \mathbb{N} \to \mathbb{N}$ defined by by $f(n) = a_n$ is primitive recursive.